SIMPLE SIMULATIONS OF MAGNETOHYDRODYNAMICS IN STAR FORMATIONS

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Abstract

Attempts have been made to implement simple simulations of MHD in star formations. Some simple and tractable numerical method is utilized to make the simulation more realistic and physically acceptable. Detailed coding are given in the Appendix.

Keywords: MHD, HFMWDs, simulation, white dwarfs

Introduction

The magnetohydrodynamics is the study of the magnetic properties and behavior of electrically conducting fluids. Examples of such magnetofluids include plasmas, liquid metals, salt water, and electrolytes. The word "magnetohydrodynamics" is derived from magneto-meaning magnetic field, hydro-meaning water, and dynamics meaning movement. The field of MHD was initiated by Hannes Alfven, for which he received the nobel Prize in Physics in 1970.

The fundamental concept behind MHD is that magnetic fields can induce currents in a moving conductive fluid, which in turn polarizes the fluid and reciprocally changes the magnetic field itself. The set of equations that describe MHD are a combination of the Navier-Strokes equations of fluid dynamics and Maxwell's equations of electromagnetism. These differential equations must be solved simultaneously, either analytically or numerically.

The Equations of Ideal MHD

The equations of ideal MHD describe the movement of a compressible conducting fluid subject to magnetic fields. In ideal MHD all dissipative processes are neglected, meaning that the fluid possesses no viscosity and its conductivity is assumed to be infinite. The ideal MHD equations (Strang, 1968).

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$$\frac{\partial \rho}{\partial t} = \nabla.(\rho v) = 0 \tag{1}$$

$$\frac{\partial \rho v}{\partial t} + \nabla.(v\rho v - bb) + \nabla P_{tot} = -\rho \nabla \phi \tag{2}$$

$$\frac{\partial E}{\partial t} + \nabla.[(E + P_{tot})v - b(v.b)] = -\rho v.\ \nabla \phi \tag{3}$$

$$\frac{\partial b}{\partial t} - \nabla \times (v \times b) = 0 \tag{4}$$

expressing the conservation of mass, momentum, energy and magnetic flux, respectively. Here ρ is the mass density, v the velocity and $E = \rho e + \frac{\rho}{2} v^2 + \frac{b^2}{2}$ the total energy density being, the sum of internal, kinetic and magnetic energy densities. The magnetic field is given by $B = \sqrt{4\pi} b$ and $P_{tot} = p + \frac{b^2}{2}$ is the total pressure, being the sum of the gas pressure and the magnetic pressure. For the equation of state (EoS) one assumes an ideal gas law

$$p = \rho e(\gamma - 1), \tag{5}$$

where γ is the ratio of specific heats. General EoSs can be included, which is especially important for the simulation of CCSN where the EoS has to describe stellar matter in the very broad range of conditions prevailing during this event. The right hand side of the momentum and energy conservation equations detail the effect of gravitational forces onto the conserved variables.

The MHD equation (1) to equation (4) conserve the divergence of the magnetic field so that an initial condition

$$\nabla.\ b = 0 \tag{6}$$

remains true, consistent with the physical fact that magnetic monopoles have never been observed.

Numerical Solution of the MHD Equations

The MHD equation form a set of non-linear hyperbolic equations, which can in general only be solved by numerical means. Before we start describing the individual solution operators, we first introduce our notation. We discretise time into discrete steps Δt^n and space into finite volumes or cells $V_{i,j,k}$ where n labels the different time levels and the triple (i,j,k) denotes a particular cell. The vector $u=(\rho,\rho v_x,\rho v_y,E)^T$ denotes the conserved fluid variables. The solution vector $u^n{}_{i,j,k}$ contains the spatially averaged values of the conserved variables at time t in cell $V_{i,j,k}$

$$u_{i,j,k}=\frac{1}{V_{i,j,k}}\int_{V_{i,j,k}} u(x,t)dxdydz, \tag{7}$$

where the cell volume $V_{i,j,k}=\Delta x\Delta y\Delta z$ is given by the assumed constant cell dimensions $\Delta x=x_{i+1/2}-x_{i-1/2},\Delta=y_{i+1/2}-y_{i-1/2},\Delta z=z_{i+1/2}-z_{i-1/2}$. Half-integer indices denote the intercell boundary. Further we define the cell face averaged magnetic field components at time t by

$$(b_x)_{i+1/2,j,k}=\frac{1}{S_{i+1/2,j,k}}\int_{S_{i+1/2,j,k}} b_x(x,t)dydz \tag{8}$$

$$(b_y)_{i,j+1/2,k}=\frac{1}{S_{i,j+1/2,k}}\int_{S_{i,j+1/2,k}} b_y(x,t)dxdz \tag{9}$$

$$(b_z)_{i,j,k+1/2}=\frac{1}{S_{i,j,k+1/2}}\int_{S_{i,j,k+1/2}} b_z(x,t)dxdy \tag{10}$$

where $S_{i+1/2,j,k}=\Delta y\Delta z$ denotes the cell face of cell $V_{i,j,k}$ located at $x_{i+1/2}$ and spanned by the zone increments Δy and Δz.

In an operator-split scheme the solution algorithm to the ideal MHD equations can be summarized as

$$u^{n+2} = L_{forward}\ L_{backward}\ u^{n}, \tag{11}$$

where

$$L_{forward} = L_x(\Delta t)B_x^{yz}(\Delta t)L_y(\Delta t)B_y^{xz}(\Delta t)L_z(\Delta t)B_z^{xy}(\Delta t) \tag{12}$$

$$L_{backward} = L_z(\Delta t)B_z^{xy}(\Delta t)L_y(\Delta t)B_y^{zx}(\Delta t)L_{zx}(\Delta t)B_x^{yz}(\Delta t) \tag{13}$$

are the forward and backward operator for one time step. The operators $L_{x,y,z}$ evolve the fluid and account for the source terms, while the B operators evolve the magnetic field. If the individual operators are second order accurate, then the application of the forward followed by the backward operator is second order accurate in time (Landau & Lifschitz, 1991). The numerical solution algorithm to the MHD equations is explicit. Therefore we impose the following time step

$$\Delta t^{n} = k.\min_{i,j,k}\left(\frac{\Delta x}{C_{i,j,k}^{n,x}}, \frac{\Delta y}{C_{i,j,k}^{n,y}}, \frac{\Delta z}{C_{i,j,k}^{n,z}}\right), \tag{14}$$

where

$$C_{i,j,k}^{n,d} = \max(v_{d,i,j,k}^{n,d} + c_{Fi,j,k}^{n}) \tag{15}$$

is the maximum speed at which information can travel in the whole computational domain in direction $d = x, y, z$ being the sum of the velocity component in d and the speed of the fast magnetosonic waves c_F. We typically set the CFL number k to 0.75.

Solving the Fluid MHD Equations

The evolution of the fluid variables u in the x-direction, we neglect the source terms from gravity. During this process the magnetic field is held

constant and interpolated to cell centers. Then mass, momentum and energy conservation in x-direction can be written as

$$\frac{\partial u}{\partial t}+\frac{\partial F}{\partial x}=0, \tag{16}$$

where

$$F=\begin{bmatrix} \rho v_x \\ \rho v_x^2+P_{tot}+b_x^2 \\ \rho v_x v_y-b_x b_y \\ \rho v_x v_z-b_x b_z \\ (E+P_{tot})v_x-b_x b.v \end{bmatrix} \tag{17}$$

is the flux vector.

Integrating equation (16) over a cell $V_{i,j,k}$ gives

$$\frac{\partial u_{i,j,k}}{\partial t}+\frac{1}{\Delta x}(F_{i+1/2,j,k}-F_{i-1/2,j,k})=0, \tag{18}$$

where the definition of the cell averaged values of equation (7) has been substituted and Gauss' theorem has been used. The numerical flux $F_{i+1/2,j,k}$ represents an average flux of the conserved quantities through the surface $S_{i+1/2,j,k}$

$$F_{i+1/2,j,k}=\frac{1}{S_{i+1/2,j,k}}\int_{S_{i+1/2,j,k}} F(x,t)dydz \tag{19}$$

at given time t. Equation (18) is a semi-discrete conservative scheme for the conservation law of equation (16). In the following we focus on obtaining the numerical fluxes in a stable and accurate manner. Time integration of the ordinary differential equation (18) will be addressed later in this subsection.

Many schemes for the stable and accurate computation of the numerical fluxes have been devised in the literature. Godunov type methods

achieve this by solving either exact or approximate Riemann problems at cell interfaces (Godunov, 1959, Laney, 1998 & Toro, 1997). Through solving the Riemann problem, these methods ensure an upwind discretisation of the conservation law and hence achieve causal consistency. Due to the difficulty of solving the Riemann problem in the ideal MHD case, the algorithm of (Pen, Arras, & Wong, 2003) uses the relaxation scheme of (Jin & Xin, 1995). For detailed information on these type of methods we refer to (Jin & Xin, 1995, LeVeque & Pelanti, 2001). The idea of the relaxation scheme is to replace a system like equation (18) by a larger system

$$\frac{\partial u}{\partial t}+\frac{\partial w}{\partial x}=0, \tag{20}$$

$$\frac{\partial w}{\partial t}+D^2\frac{\partial u}{\partial x}=\frac{1}{\varepsilon}(F(u)-w), \tag{21}$$

called the relaxation system. Here, the relaxation rate ε is a small positive parameter and D^2 is a positive definite matrix. For small relaxation rates, system of equation (20) rapidly relaxes to the local equilibrium defined by $w=F(u)$. A necessary condition for solutions of the relaxation system of equation (20) to converge in the small ε limit to solutions of the original system of equation (20) is that the characteristic speeds of the hyperbolic part of equation (20) are at least as large or larger than the characteristic speeds in system of equation (16). This is the so-called subcharacteristic condition.

As (Jin & Xin,1995) we choose $D=d.\ I$ to be a diagonal matrix. In order to fulfill the subcharacteristic condition the diagonal element d or the so-called freezing speed is chosen to be

$$d=|v_x|+c_{F,} \tag{22}$$

where c_F is the speed of the fast magnetosonic waves, i.e. the fastest wave propagation speed supported by the equations of ideal MHD.

The key point in the relaxation system is that in the local equilibrium limit it has a very simple characteristic structure

$$\frac{\partial}{\partial t}(w+Du)+D\frac{\partial}{\partial x}(w+Du)=0, \tag{23}$$

$$\frac{\partial}{\partial t}(w-Du)-D\frac{\partial}{\partial x}(w-Du)=0, \tag{24}$$

where $w\pm Du$ are then the characteristic variables. They travel with the "frozen" characteristic speeds $\pm D$ respectively.

System of equation (23), (24) can be easily recast into an equation for u and w. However, we are practically only interested in that for u

$$\frac{\partial u}{\partial t}+\frac{\partial F^{+}}{\partial x}+\frac{\partial F^{-}}{\partial x}=0, \tag{25}$$

where $F^{+}=(w+Du)/2$ denotes the right travelling waves and $F^{-}=(w-Du)/2$ the left travelling waves in the x-direction. In the following we shall drop the indices of the other directions. Since this defines an upwind direction for each wave component, a first order upwind scheme results from choosing $F^{+}_{i+1/2}=F^{+}_{i}$ and $F^{-}_{i+1/2}=F^{-}_{i+1}$. In this case, the total flux at the cell interfaces is readily evaluated to become

$$F_{i+1/2}=F^{+}_{i+1/2}+F^{-}_{i+1/2}=\frac{1}{2}(F_i+F_{i+1})-\frac{1}{2}D(u_{i+1}-u_i)\ , \tag{26}$$

where $F_i=w_i=F(u_i).$ For D we use the freezing speed

$$d=\max(d_i,d_{i+1}) \tag{27}$$

in order to satisfy the subcharacteristic condition.

Conclusion Remarks

In this paper, simple simulations for magnetohydrodynamics (MHD) have been presented using simple iteration and simple mathematica coding to study the numerical nature of MHD equations and it is simply observed that some of the numerical simulations give interesting 2D and 3D graphics which show the physical nature of the equations.

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Appendix

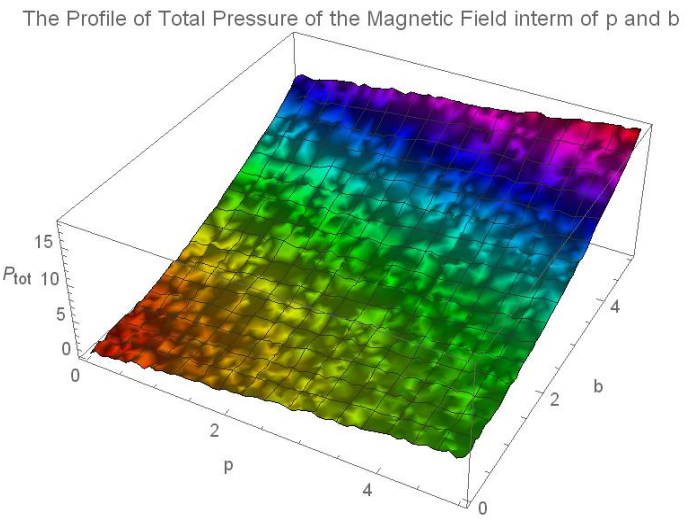

Figure 1: The profile of total pressure of the magnetic field in term of p and b.

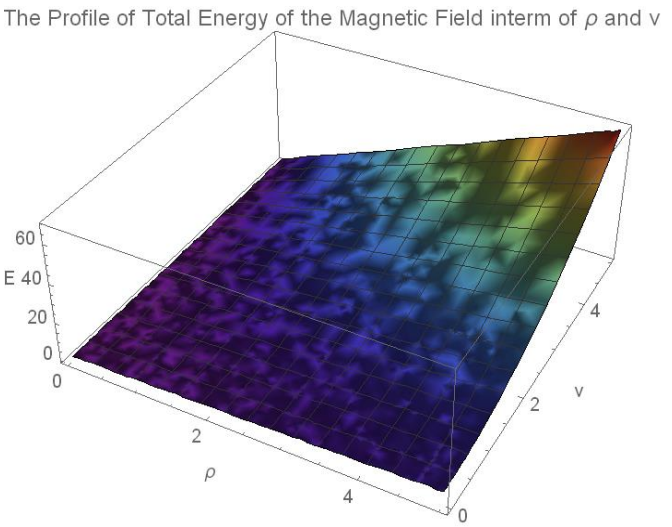

Figure 2: The profile of total energy of the magnetic field in term of ρ and v.

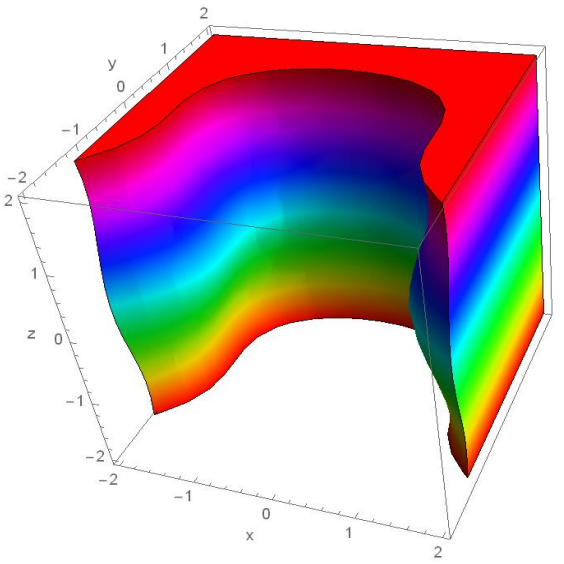

Figure 3: The profile of solution vector $u^n{}_{i,j,k}$ contains the spatially averaged values.